Computing Pure Nash Equilibria in Symmetric Action Graph Games

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Outline

1. Action Graph Games
2. Computing Pure Nash Equilibria
3. Computing Pure Equilibira in Symmetric AGGs
4. Algorithm
5. Conclusions

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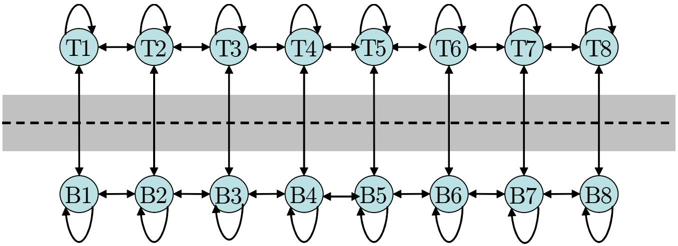
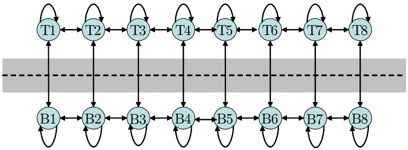
Example: Location Game

- each of n agents wants to open a business
- actions: choosing locations
- utility: depends on
  - the location chosen
  - number of agents choosing the same location
  - numbers of agents choosing each of the adjacent locations



Game on a graph



- This can be modeled as a game played on a directed graph:
  - each player has a token to put on one of the nodes;
  - each player's utility depends on:
    - the node chosen
    - configuration of tokens over neighboring nodes
- Action Graph Games (Bhat & Leyton-Brown 2004, Jiang & Leyton-Brown 2006)
  - fully expressive, compact representation of games
  - exploits anonymity, context specific independence

Definitions

Definition (action graph)

An action graph is a tuple $(\mathcal{A}, E)$, where $\mathcal{A}$ is a set of nodes corresponding to *distinct actions* and E is a set of directed edges.

- Each agent i's set of available actions: $A_i \subseteq \mathcal{A}$
- Neighborhood of node α: $\nu(\alpha) \equiv \{\alpha' \in \mathcal{A} | (\alpha', \alpha) \in E\}$

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Definition (configuration)

A **configuration** c is an $|\mathcal{A}|$-tuple of integers $(c[\alpha])_{\alpha \in \mathcal{A}}$. $c[\alpha]$ is the number of agents who chose the action $\alpha \in \mathcal{A}$. For a subset of actions $X \subset \mathcal{A}$, let $c[X]$ denote the restriction of c to X. Let $C[X]$ denote the set of restricted configurations over X.

Action Graph Games

Definition (Action Graph Game (AGG))

An action graph game Γ is a tuple $\langle N, (A_i)_{i\in N}, G, u\rangle$ where

- N is the set of agents
- A_i is agent i's set of actions
- $G = (\mathcal{A}, E)$ is the action graph, where $\mathcal{A} = \bigcup_{i\in N} A_i$ is the set of distinct actions
- $u = (u^\alpha)_{\alpha\in\mathcal{A}}$, where $u^\alpha : C[\nu(\alpha)] \mapsto \mathbb{R}$

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Definition (symmetric AGG)

An AGG is symmetric if all players have identical action sets, i.e. if $A_i = \mathcal{A}$ for all i.

AGG Properties

- AGGs are fully expressive
- Symmetric AGGs can represent arbitrary symmetric games
- Representation size $\|\Gamma\|$ is polynomial if the in-degree $\mathcal{I}$ of G is bounded by a constant
- Any graphical game (Kearns, Littman & Singh 2001) can be encoded as an AGG of the same space complexity.
- AGG can be exponentially smaller than the equivalent graphical game & normal form representations.

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Pure Nash Equilibria

Action profile: $\mathbf{a} = (a_1, \ldots, a_n)$

Definition (pure Nash equilibrium)

An action profile $\mathbf{a}$ is a pure Nash equilibrium of the game Γ if for all $i \in N$, a_i is a best response to a_{-i} (i.e. for all $a_i' \in A_i$, $u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})$).

- not guaranteed to exist
- often more interesting than mixed Nash equilibria

Complexity of Finding Pure Equilibria

Checking every action profile:

- linear time in normal form size
- worst-case exponential time in AGG size

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Checking every action profile:

- linear time in normal form size
- worst-case exponential time in AGG size

Consider the restriction to symmetric AGGs.

Theorem (Conitzer, personal communication; also proven independently in (Daskalakis *et al.* 2008))

The problem of determining whether a pure Nash equilibrium exists in a symmetric AGG is NP-complete, even when the in-degree of the action graph is at most 3.

Our Contribution

We provide an algorithm that is tractable for symmetric AGGs with bounded treewidth

- the algorithm can also be applied to other settings

Specifically, we propose a dynamic programming approach:

- partition action graph into subgraphs (via tree decomposition)
- construct equilibria of the game from equilibria of games played on subgraphs

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We provide an algorithm that is tractable for symmetric AGGs with bounded treewidth

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Related Work:

- finding pure equilibria in graphical games
  - (Gottlob, Greco, & Scarcello 2003) and (Daskalakis & Papadimitriou 2006)
- finding pure equilibria in simple congestion games
  - (Ieong, McGrew, Nudelman, Shoham, & Sun 2005)

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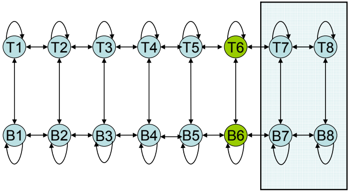
Restricted Game

To derive an algorithm that builds up from partial solutions, we must define the concept of a restricted game

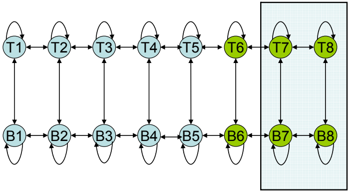
- game played by a subset of players: $n' \leq n$
- actions restricted to $R \subseteq \mathcal{A}$
- utility functions same as in original AGG
  - need to specify configuration of neighboring nodes not in R



- *restricted game* $\Gamma(n', R, c[\nu(R)])$

Partial Solution

We want to use equilibria of restricted games as building blocks



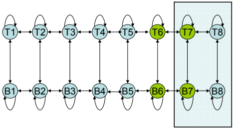
Definition (partial solution)

A partial solution on a restricted game $\Gamma(n', X, c[\nu(X)])$ is a configuration $c[X \cup \nu(X)]$ such that $c[X]$ is a pure NE of Γ.

Extending partial solutions

- Problem: combining two partial solutions on two non-overlapping restricted games does not necessarily produce an equilibrium of the combined game
  - configurations may be inconsistent, or
  - player might profitably deviate from playing in one restricted game to another
- keeping all partial solutions: impractical as sizes of restricted games grow
- we would like sufficient statistics that summarize partial solutions as compactly as possible

Sufficient statistic



Sufficient Statistic: a tuple consisting of

1. configuration over
   - outside neighbours: $\nu(X)$
   - inside nodes that are neighbors of outside nodes: $\nu(\overline{X})$
2. number of agents playing in X
3. U_w, utility of the worst-off player in $X \setminus \nu(\overline{X})$.
4. U_b, best utility an outside player can get by playing in $X \setminus \nu(\overline{X})$.

Number of distinct tuples: polynomial for action graphs of bounded treewidth

Combining sufficient statistics

Given two sets of such tuples, summarizing partial solutions on $X, Y \subset \mathcal{A}$, we can compute the set of sufficient statistics for the combined restricted game $X \cup Y$

- start with all consistent configurations
  - analogous to database join of the two sets of tuples
- discard those with profitable $X \to Y$ deviations (& vice versa)
  - easy: discard when U_w from X is worse than U_b from Y
  - trickier: checking deviations from $X \cap \nu(Y)$ to $\nu(\overline{Y})$
    - utilities in $\nu(\overline{Y})$ change when $c[\nu(Y)]$ changes, so checking these deviations is more costly
    - solution: augment our sufficient statistics to keep track of the configuration of the neighborhood of $\nu(\overline{Y})$, in order to compute these utilities on the fly
    - luckily, for graphs of bounded treewidth, this implies storing a small amount of additional information
  - overall: all profitable deviations can be discarded efficiently

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Algorithm

1. Construct the primal graph of the action graph.
2. Build a tree decomposition of this primal graph.
3. Partition the AGG according to the tree decomposition.
4. Find all sufficient statistics[1] corresponding to partial solutions of games restricted to each partition.
5. Working up the tree, combine adjacent nodes together.
6. When root is reached, return whether the game has a PSNE.

[1] Augment sufficient statistics to include configurations over additional actions that belong to the decomposition's tree node that is closest to the root.

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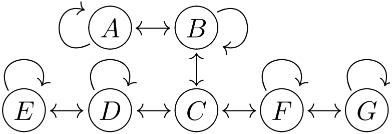
Theorem

For symmetric AGGs with bounded treewidth, our algorithm determines existence of pure Nash equilibria in polynomial time.

Recover a PSNE from the SS's: downwards pass on the tree

[1] Augment sufficient statistics to include configurations over additional actions that belong to the decomposition's tree node that is closest to the root.

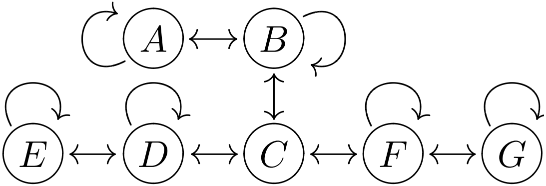
An Example



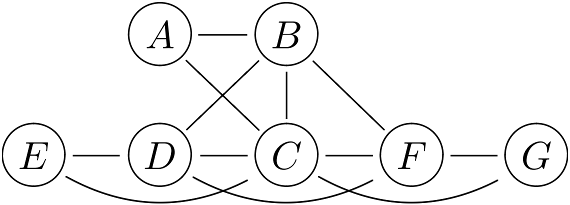
- Two players
- Utility functions:
  - start with payoff of 0
  - $+1$ reward if playing action F or D
  - -2 penalty if another player selected an action with an incoming edge
    - For C, this means a neighboring action (since C does not have a self-edge)
    - Otherwise, this means the same or a neighboring action
- Pure Nash equilibria:
  - One player chooses D, the other chooses F
  - Both players choose C

1. Construct Primal Graph

Action graph:



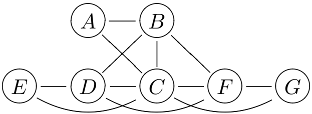
Primal graph: make each neighborhood a clique

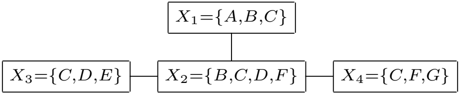


2. Construct Tree Decomposition

A tree where each node is labeled with one or more nodes from the primal graph, where

- every label is used at least once
- for every edge in the primal graph from α_1 to α_2, there is a node in the tree labeled with both α_1 and α_2
- if a label occurs in two nodes x_1, x_2 in the tree, it also occurs on all paths between x_1 and x_2.



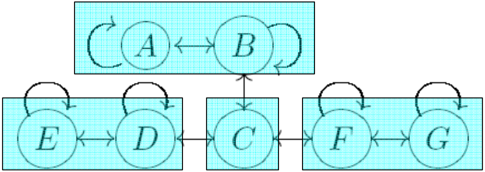


If treewidth of the AGG is bounded by a constant, the primal graph's tree decomposition can be computed in polynomial time.

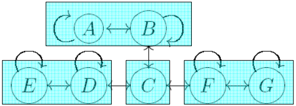
3. Partition the AGG According to the Tree Decomposition

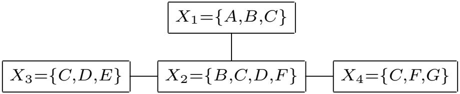
By construction: for each node α in the action graph, there always exists a tree node in the decomposition of the primal graph that contains α and its neighbors in the action graph.

The tree decomposition therefore induces the following partition on the AGG:



4. Compute Sufficient Statistics for Partial Solutions on Each Partition





For restricted game on $\{C\}$:

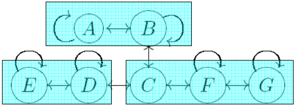
n'	$c[B,C,D,F]$	$U_w(\emptyset)$	$U_b(\emptyset)$
0	0,0,0,0	∞	$-\infty$
0	1,0,0,0	∞	$-\infty$
$\cdots$	$\cdots$	∞	$-\infty$
1	0,1,0,0	∞	$-\infty$
1	1,1,0,0	∞	$-\infty$
$\cdots$	$\cdots$	∞	$-\infty$
2	0,2,0,0	∞	$-\infty$

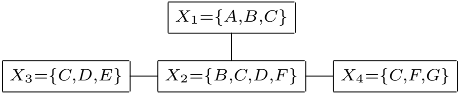
For restricted game on $\{F,G\}$:

n'	$c[C,F,G]$	$U_w(G)$	$U_b(G)$
0	0,0,0	∞	0
0	1,0,0	∞	0
0	2,0,0	∞	0
1	0,1,0	∞	-2
1	1,0,1	0	-2
2	0,1,1	-2	$-\infty$

5. Working up the Tree, Combine Restricted Games

Combine restricted games in bottom-up order: from leaves to root.





Combine {C} and {F,G} to create table for restricted game on {C,F,G}:

n'	$c[B,C,D,F]$	$U_w(G)$	$U_b(G)$
0	0,0,0,0	∞	0
0	1,0,0,0	∞	0
...	...	∞	0
1	0,0,0,1	∞	-2
1	1,0,0,1	∞	-2
1	0,0,1,1	∞	-2
2	0,1,0,0	0	$-\infty$
2	0,2,0,0	∞	$-\infty$

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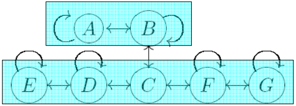

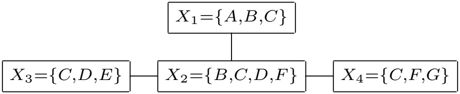

Combine {D,E} and {C,F,G} to create table for {C,D,E,F,G}:

n'	$c[B,C,D,F]$	$U_w(E,G)$	$U_b(E,G)$
0	0,0,0,0	∞	0
0	1,0,0,0	∞	0
0	2,0,0,0	∞	0
1	0,0,1,0	∞	0
1	1,0,1,0	∞	0
1	0,0,0,1	∞	0
1	1,0,0,1	∞	0
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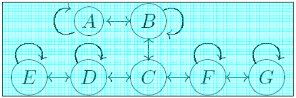

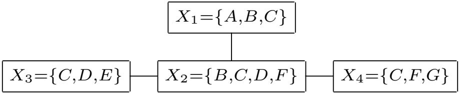

Combine {A,B} and {C,D,E,F,G}:

n'	$c[A,B,C]$	$U_w(D,E,F,G)$	$U_b(D,E,F,G)$
2	0,0,0	1	$-\infty$
2	0,0,2	∞	$-\infty$

6. Top-Down Pass to Compute PNSE

n'	$c[A,B,C]$	$U_w(D,E,F,G)$	$U_b(D,E,F,G)$
2	0,0,0	1	$-\infty$
2	0,0,2	∞	$-\infty$

To compute a PSNE, start from the root and work down. At each node, pick a row from the table of sufficient statistics that is consistent with earlier picks.

- If we start with row 1, we select an equilibrium in which one player chooses D, one player chooses F
- If we start with row 2, we select an equilibrium in which both players choose C

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Conclusions & Beyond Symmetric AGGs

- dynamic programming approach for computing pure equilibria in AGGs
- poly-time algorithm for symmetric AGGs with bounded treewidth
- our approach can be extended to general AGGs
  - different set of sufficient statistics
    - when the game is k-symmetric (i.e. has k distinct action sets), use k-configuration (k-tuple of configurations, one for each equivalence class of players), and similarly use k-tuples of U_w, U_b
    - for subgraphs in which only k' of the k classes of players participate, only need to keep track of the sufficient statistics for those k' classes.
  - related algorithms for graphical games (Daskalakis & Papadimitriou 2006) and simple congestion games (Ieong et al 2005) become special cases of our approach